equation:

$\Omega |v\rangle = \omega |v\rangle \Rightarrow (\Omega - \underbrace{\omega I}_{\text{identity operator}}) |v\rangle = 0$

In n dimensional vector space $\mathbb{V}^n$, matrix representation of Ω (and I) is a $n \times n$ matrix. The characteristic equation has nontrivial solutions ($|v\rangle \neq 0$) only if;

$\det(\Omega - \omega I) = 0$

But the left hand side is a polynomial of nth order in ω. The eigenvalue problem then comes down to finding the solutions of a nth degree equation.

Such a polynomial always has n roots, but not necessarily real or distinct. Therefore, in general, an operator Ω may have no eigenvectors. However, any Hermitian or unitary operator in $\mathbb{V}^n(\mathbb{C})$ has n eigenvectors.

As we showed, the eigenvectors of Hermitian and unitary operators are orthogonal. Hence they form an orthonormal basis (upon normalization of eigenvectors). In this basis the matrix representation of the operator is a diagonal matrix.

For two commuting Hermitian operators, there exists (at least) a basis of common eigenvectors. Matrix representations of both operators are diagonal in this basis. The operators can thus be simultaneously diagonalized.

Functions of Operators:

Consider a function $f(x)$ that can be written as a power series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

We define the same function of an operator Ω as:

$$f(\Omega) = \sum_{n=0}^{\infty} a_n \Omega^n$$

For example:

$$e^{\Omega} = \sum_{n=0}^{\infty} \frac{\Omega^n}{n!}$$

Once we know the matrix representation of Ω, that of $f(\Omega)$ will be known too.

Operators themselves can be functions of parameters. For example:

$$\theta(\lambda) = e^{\lambda \Omega}$$

Derivative of an operator with respect to a parameter is defined in the usual way, and can be calculated using the power series expansion:

$$\frac{d}{d\lambda} e^{\lambda\Omega} = \frac{d}{d\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n \Omega^n}{n!} = \sum_{n=1} \frac{n\lambda^{n-1}\Omega^n}{n!} = \Omega e^{\lambda\Omega}$$

Since $\left[\Omega, e^{\lambda\Omega}\right] = 0$, the order does not matter:

$$\frac{d}{d\lambda} e^{\lambda\Omega} = \Omega e^{\lambda\Omega} = e^{\lambda\Omega}\Omega$$

However, in general, one has to be careful with ordering. for example:

$$\frac{d}{d\lambda}\left(e^{\lambda\Omega} e^{\lambda\Lambda}\right) = \Omega e^{\lambda\Omega} e^{\lambda\Lambda} + e^{\lambda\Omega} \Lambda e^{\lambda\Lambda}$$

In general, we define:

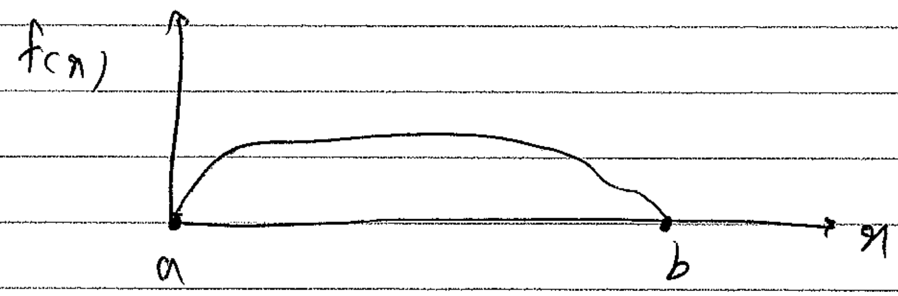
$$\frac{d}{d\lambda}\left(\Omega_1(\lambda)\Omega_2(\lambda)\right) = \frac{d\Omega_1(\lambda)}{d\lambda}\Omega_2(\lambda) + \Omega_1(\lambda)\frac{d\Omega_2(\lambda)}{d\lambda}$$

Note that for numbers it does not matter:

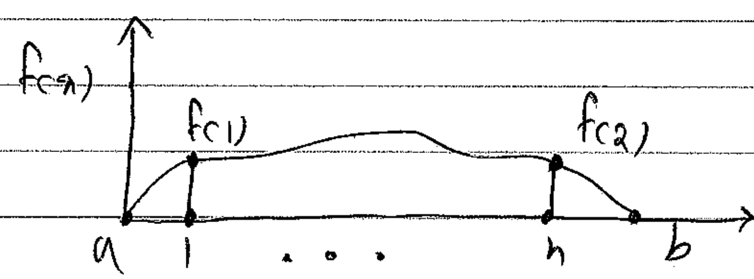
$$\frac{d}{dt}\left(f_1(\lambda) f_2(\lambda)\right) = \frac{df_1(\lambda)}{d\lambda} f_2(\lambda) + \overbrace{f_1(\lambda)\frac{df_2(\lambda)}{d\lambda}} \rightarrow \frac{df_2(\lambda)}{d\lambda} f_1(\lambda)$$

Infinite Dimensional Vector Spaces:

Consider a function $f(x)$ defined in the interval $a \leq x \leq b$.



An approximate specification of $f(x)$ is given by its value at n points between a and b:



$$f(x) \longleftrightarrow |f\rangle = \begin{bmatrix} f(1) \\ \vdots \\ \vdots \\ f(n) \end{bmatrix} = f(1) \underset{|1\rangle}{\begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}} + \cdots + f(n) \underset{|n\rangle}{\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}}$$

We can define unit vectors $|i\rangle$ where only the i-th component is nonzero (1). $\{|1\rangle, \dots, |n\rangle\}$ form an orthonormal basis.

For an exact specification of $f(x)$ we need to know its value at all points between a and b, i.e. $n \to \infty$.

Then we will have vector $|x\rangle$ where:

$\langle x' | x\rangle = \delta(x - x')$ Dirac delta function

$f(x) \leftrightarrow |f\rangle$ (a function is a vector in Hilbert space)

$f(x) = \langle x | f\rangle$

The inner product of two vector is defined as:

$\langle f | g\rangle = \int \langle f | x\rangle \langle x | g\rangle \, dx = \int f^*(x) \, g(x) \, dx$

Note that $\int |x\rangle \langle x| \, dx = I$, as $\sum_i |i\rangle\langle i|$ is in the finite dimensional case.

Operator X is defined as:

$X|x\rangle = x|x\rangle$ (X is Hermitian)

The derivative of a function is obtained by acting the differential operator D:

$$D|f\rangle = \left|\frac{df}{dx}\right\rangle$$

$$\frac{df(x)}{dx} = \langle x|D|f\rangle$$

$$\langle x|D|f\rangle = \int \langle x|D|x'\rangle\langle x'|f\rangle\, dx' = \int \langle x|D|x'\rangle f(x')\, dx' \Rightarrow \langle x|D|x'\rangle = \delta'(x-x') = \delta(x-x')\frac{d}{dx'}$$

We define the operator K as:

$$K = -iD$$

As we will see, K is a Hermitian operator.

The eigenvectors of K are:

$$K|k\rangle = k|k\rangle$$

$$\langle x|K|k\rangle = k\langle x|k\rangle = \int \langle x|K|x'\rangle\langle x'|k\rangle\, dx'$$

$$\Rightarrow -i\frac{d}{dx}\langle x|k\rangle = k\,\langle x|k\rangle \Rightarrow \langle x|k\rangle = A\,e^{ikx}$$

$$\psi_k(x) = \langle x|k\rangle = \frac{1}{\sqrt{2\pi}}\,e^{ikx}$$

$A = \frac{1}{\sqrt{2\pi}}$ we choose

This eigenvector is normalized to delta function.

Here we focus of $k \in R$ for the reason that becomes clear shortly.

Note that:

$$\langle k'|k\rangle = \int \langle k'|x\rangle\langle x|k\rangle\,dx = \frac{1}{2\pi}\int e^{ikx}\,e^{-ik'x}\,dx = \delta(k'-k)$$

Also:

$$\int |k\rangle\langle k|\,dk = \iiint |x'\rangle\langle x'|k\rangle\langle k|x\rangle\langle x|\,dx\,dx'\,dk$$

$$= \int \frac{1}{2\pi}e^{ik(x'-x)}\,|x'\rangle\langle x|\,dx\,dx' = \int \delta(x'-x)\,|x'\rangle\langle x|\,dx\,dx' = \int |x\rangle\langle x|\,dx = I$$